Exercise 16

Use the *modified decomposition method* to solve the following Volterra integral equations:

$$u(x) = x^3 - x^5 + 5\frac{1}{10}\int_0^x tu(t)\,dt$$

[TYPO: In order to get the answer at the back of the book, this factor of 1/10 should not be here.]

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\sum_{n=0}^{\infty} u_n(x) = x^3 - x^5 + 5\int_0^x t\sum_{n=0}^{\infty} u_n(t)\,dt$$

$$u_0(x) + u_1(x) + u_2(x) + \cdots = x^3 + (-x^5) + 5\int_0^x t[u_0(t) + u_1(t) + \cdots]\,dt$$

$$u_0(x) + u_1(x) + u_2(x) + \cdots = \underbrace{x^3}_{u_0(x)} + \underbrace{(-x^5) + 5\int_0^x tu_0(t)\,dt}_{u_1(x)} + \underbrace{5\int_0^x tu_1(t)\,dt}_{u_2(x)} + \cdots$$

Grouping the terms as we have makes it so that the series terminates early.

$$u_0(x) = x^3$$

$$u_1(x) = (-x^5) + 5\int_0^x tu_0(t)\,dt = (-x^5) + x^5 = 0$$

$$u_2(x) = 5\int_0^x tu_1(t)\,dt = 0$$

$$\vdots$$

$$u_n(x) = 5\int_0^x tu_{n-1}(t)\,dt = 0, \quad n > 2$$

Therefore,

$$u(x) = x^3.$$